MULTIPLIERS ON SEMI-SIMPLE Fre′chet ALGEBRA

M. Azram
Faculty of Science, DOMAS, SQU, Muscat 123, OMAN.
E.mail: azram@squ.edu.om, azram50@hotmail.com

ABSTRACT**:** *This paper is devoted to establish some fundamentally important results on a commutative semi-simple Fre′chet algebra A. Equality such as* $\Phi_M(A)=\Phi_o(A)\cap M(A)=\Phi_+(A)\cap M(A)=\Phi_-(A)\cap M(A)=\Phi(A)\cap M(A)$ *has been established among the class of Fredholm operators with respect to two sided ideals and upper and lower semi-Fredholm operators. It has also been shown that each semi-Fredholm multiplier is a product of an idempotent and an invertible multiplier. Finally we have shown that if* ΔA *has no isolated point then TϵM(A) is semi-Fredholm iff T is invertible.*

Keywords: Commutative semi simple Fréchet algebra, Banach algebras, topological algebra, locally convex algebra. Fredholm operator.

INTRODUCTION

Topological algebra is a natural generalization of Banach algebra. The concept of multipliers on Banach algebras was firstly introduced by S. Helgason [1]. The general theory of multipliers have been studied quite extensively in the context of Banach algebra without order by F.T. Birtel [2], J. K. Wang [3], R. Larsen [4] etc.. T. Husain [5] has generalized the notion of a multiplier on a topological algebra without order. N. Mohammad [6,7] has studied the spectral properties of multipliers and compact multipliers on topological algebras. L. A. Khan [8] has studied the double multipliers on topological algebras. Recently H. Render [9] has introduced the study of multipliers in the framework of vector spaces of holomorphic functions. Azram [10.11.12] has studied topological algebra via inner product and multipliers in the frame work of Fre′chet algebra. In view of applications and recent developments in the theory of topological algebras, it is important to study multipliers on more generalized topological algebras. As an example, Quantum groups are the outcome of the study of multipliers on Hopf algebra. Multipliers have immediate applications in many areas of mathematics, such as, Harmonic analysis, Differential geometry, Representation theory, Field theory, optimal control, Quantum Mechanics Statistical Mechanics and consequently in engineering, etc. As an application of control theory, one may consider the areas such as scheduling and control of engineering devices, aerospace engineering, maximum orbit transfer problem, navigation, mobile robotics and automated vehicles, etc. Solid state engineering requires the concepts of quantum mechanics. Distribution theory, propagation of heat, robotics, image analysis and decomposition of electromagnetic waves, etc require the concept of harmonic analysis. Differential geometry by means of tensor calculus is an application in engineering. It is worth observing that not much work has been done in the setting of topological algebra.In view of applications and recent developments in the theory of topological algebras, it is important to study multipliers on more generalized topological algebras. As an example, Quantum groups are the outcome of the study of multipliers on Hopf algebra. David van Dantzig [13] has used the word "Topological Algebra' very first time in his Ph.D. thesis and later in a whole series of papers. Later on it also appeared in the literature by R. Arens [14]. Fundamental results, although at the same time but separately, were published by R. Arens [14] and E.A. Michael [15]. The study of these algebras helped to investigate the non-normable behaviours in mathematics and physics.

This paper is devoted to establish some fundamentally important results on a commutative semi-simple Fre′chet algebra A. Equality such as,

$$\Phi_M(A)=\Phi_o(A)\cap M(A)=\Phi_+(A)\cap M(A)=\Phi_-(A)\cap M(A)=\Phi(A)\cap M(A)$$

has been established among the class of Fredholm operators with respect to two sided ideals and upper and lower semi-Fredholm operators. Consequently, showing that each semi-Fredholm multiplier is a product of an idempotent and an invertible multiplier. Finally we have shown that if ΔA has no isolated point then TϵM(A) is semi-Fredholm iff T is invertible.

MATERIAL AND METHODS

A vector space A over the complex field C is called algebra if

i. A is closed i.e., $\forall x,y\in A,\ xy\in A$
ii. $x(yz)=(xy)z=xyz\ \ \forall x,y,z\in A$
iii. $x(y+z)=xy+xz$ and $(x+y)z=xz+yz\ \ \forall x,y,z\in A$
iv. $(\lambda x)(\mu y)=(\lambda\mu)(xy)\ \ \forall x,y\in A$ and $\lambda,\mu\in C$

Algebra A with Hausdorff topology and continuous algebraic operations will be called topological algebra. A topological algebra will be called locally convex algebra if its topology is generated by the semi norms $\{p_\alpha\}_{\alpha\in I}$ such that $\forall\alpha\in I$

i. $p_\alpha(x)\ge 0$ and $p_\alpha(x)=0$ if $x=0$
ii. $p_\alpha(x+y)\le p_\alpha(x)+p_\alpha(y)\ \ \forall x,y\in A$
iii. $p_\alpha(\lambda x)=|\lambda|\,p_\alpha(x)\ \ \forall x\in A$ and $\lambda\in C$

A locally convex algebra will be called locally multiplicatively convex if $p_\alpha(xy)\le p_\alpha(x)\,p_\alpha(y)\ \ \forall\alpha\in I\ \&\ \forall x,y\in A$.

A complete metrizable locally multiplicatively convex algebra is called $Fre'chet\ a\lg ebra$. ΔA will be the set of all non-zero continuous multiplicative linear functionals on A. If A is a commutative locally multiplicatively convex algebra then; $\mathrm{rad}(A)=\{x\in A: f(x)=0\ \forall f\in\Delta A\}$. In the case of $\mathrm{rad}(A)=\{0\}$, it will be called semi simple. $soc(A)$ will be the sum of minimal ideals of A. A multiplier on A will be a mapping $T: A\to A \ni Tx.y=x.Ty\ \forall x,y\in A$ and the set of all multipliers on A will be denoted by $M(A)$. Multipliers on a semi simple algebra are linear. Multipliers on a proper (without order) complete metrizable algebra [17] and consequently on Fre′chet algebra are continuous. If A is Fre′chet algebra then $T(A)$ and $\ker(T)$ are two sided ideals of A. If A is Fréchet algebra then the range of T denoted as T(A) and ker (T) are two sided ideals of A. If A be a commutative semi simple Fréchet algebra and T is a linear bounded linear operator on A, we define $\alpha(T)=\dim\ker(T)$ and $\beta(T)=\dim(A/T(A))$. If $\alpha(T)<\infty$ and $\beta(T)<\infty$, then T is said to have a finite deficiency. Collection of all continuous linear self mapping of A will be denoted by B(A). If A is commutative Fréchet algebra and $T\in B(A)$ then T is called Fredholm operator if T has finite deficiency. The set of all Fredholm oprators on A will be denoted by $\Phi(A)$. K(A) is set of all compact linear operators on A. $K_M(A)=K(A)\cap M(A)$ and is a two sided ideal of M(A). $\Phi_M(A)$ is class of all Fredholm elements of M(A) relative to $K_M(A)$.

$\Phi_M(A)=\{T\in M(A): T \text{ is invertible in } M(A) \bmod K_M(A)\}$.

The set of upper and lower semi-Fredholm operators of A will be denoted as $\Phi_+(A)=\{T\in B(A):\alpha(T)<\infty \text{ and } T(A) \text{ is closed}\}$ and $\Phi_+(A)=\{T\in B(A):\beta(T)<\infty\}$ respectively.

Theorem 1:

If A be a commutative semi simple Fre′chet algebra with $T\in M(A)$ and $\overline{soc(A)}=A$ then

$\Phi_M(A)=\Phi_o(A)\cap M(A)$

Proof: Let

$T\in\Phi_M(A)\Rightarrow\exists S\in M(A)$ and $K\in K_M(A)\ni TS=I+K$.

M(A) is commutative so $TS=ST=I+K$.

$TS=ST=I+K\in\Phi(A)$ with index 0 [16].

Ind(T) + ind(S) = 0 [16].

$p(T)\le 1\ \forall T\in M(A)\Rightarrow\alpha(T)\le\beta(T)$ [16].

Consequently, Ind(T) = ind(S) = 0 $\Rightarrow T\in\Phi_o(A)\cap M(A)\Rightarrow$

$\Phi_M(A)\subseteq\Phi_o(A)\cap M(A)$

Conversely, let $T\in\Phi_o(A)\cap M(A)$. By Wedderburn theorem, ker(T) is sum of minimal ideals i.e. there exist idempotent e in A such thar ker(T) = e A.

Define;

$L_e: A\to A$ as $L_e x=ex=x\ \forall x\in A$. We claim that $T-L_e\in\Phi(A)\cap M(A)$

$\mathrm{ind}(T-L_e)=\mathrm{ind}(T)=0$ and $(T-L_e)xy=x(T-L_e)y\ \forall x,y\in A\Rightarrow$

$T-L_e\in M(A)$.

By [16]

$T-L_e\in\Phi(A)$ and since $\mathrm{ind}(T)=\mathrm{ind}(T-L_e)=0\Rightarrow$

$T-L_e\in\Phi_o(A)\cap M(A)$.

$\ker(T-L_e)=\{0\}\Rightarrow T-L_e$ is invertible. If S is inverse of

$T-L_e\Rightarrow TS=I+L_eS$ where $L_eS\in K_M(A)\Rightarrow T\in\Phi_M(A)$

$\therefore\ \Phi_o(A)\cap M(A)\subseteq\Phi_M(A)$.

Theorem 2:

If A be a commutative semi simpl Fre′chet algebra with $T\in M(A)$ and $\overline{soc(A)}=A$ then

$\Phi(A)\cap M(A)=\Phi_-(A)\cap M(A)$

Proof: let

$T\in\Phi_-(A)\cap M(A)\Rightarrow\beta(T)<\infty$.

$p(T)\le 1<\infty\ \forall T\in M(A)\Rightarrow\alpha(T)\le\beta(T)<\infty$ [16]. Hence, $T\in\Phi(A)\cap M(A)$.

Also $\Phi=\Phi_+\cap\Phi_-\Rightarrow\Phi\subseteq\Phi_-$.

Hence, $\Phi(A)\cap M(A)=\Phi_-(A)\cap M(A)$

Theorem 3:

If A be a commutative semi simple Fre′chet algebra with $T\in M(A)$ and $\overline{soc(A)}=A$ then

$\Phi_+(A)\cap M(A)=\Phi_-(A)\cap M(A)$

Proof: Let

$T\in\Phi_+(A)\cap M(A)\Rightarrow\alpha(T)<\infty$ and $T(A)$ is closed [16].

$T(A)+\ker(T)$ is closed and

$A=T(A)\oplus\ker(T)\Rightarrow\beta(T)\le\alpha(T)<\infty\Rightarrow$

$T\in\Phi_-(A)\cap M(A)$.

Similarly if

$T\in\Phi_-(A)\cap M(A)\Rightarrow\alpha(T)\le\beta(T)<\infty\Rightarrow$

$T\in\Phi_+(A)\cap M(A)$.

Theorem 4:

If A be a commutative semi simple Fre′chet algebra with $T\in M(A)$ and $\overline{soc(A)}=A$ then

$\Phi_M(A)=\Phi_o(A)\cap M(A)=\Phi_+(A)\cap M(A)$

$=\Phi_-(A)\cap M(A)=\Phi(A)\cap M(A)$

Proof:

$\Phi_M(A)=\Phi_o(A)\cap M(A)$ [Theorem 1] (i)

$\Phi(A)\cap M(A)=\Phi_-(A)\cap M(A)$ (ii)

$\Phi_+(A)\cap M(A)=\Phi_-(A)\cap M(A)$ [Theorem 3] (iii)

Now, let

$T\in\Phi_-(A)\cap M(A)\Rightarrow\beta(T)<\infty\Rightarrow\alpha(T)\le\beta(T)<\infty$ **[16].**

Hence, T is a Fredholm multiplier i.e.

$T\in\Phi(A)\cap M(A)\Rightarrow\Phi_-(A)\cap M(A)\subseteq\Phi(A)\cap M(A)$.

Since, $\Phi(A)=\Phi_+(A)\cap\Phi_-(A)\Rightarrow\Phi(A)\subseteq\Phi_-(A)$.

Hence, $\Phi_-(A)\cap M(A)=\Phi(A)\cap M(A)$ (iv)

$T\in\Phi_+(A)\cap M(A)\Rightarrow\beta(T)\le\alpha(T)<\infty\Rightarrow$

$T\in\Phi_o(A)\cap M(A)$.

Hence. $\Phi_o(A)\cap M(A)=\Phi_+(A)\cap M(A)$. (v)

(i)-(v) $\Rightarrow$ As required.

Consequences of theorem 3, we can have the following result;

Corollary:

If A be a commutative semi-simple $\text{Fre}'\text{chet}$ algebra with $T \in M(A)$ and $\overline{soc(A)} = A$ then each semi-Fredholm multiplier is a product of an idempotent and an invertible multiplier.

Proposition:

If A be a commutative semi simple $\text{Fre}'\text{chet}$ algebra with $T \in M(A)$ and $\overline{soc(A)} = A$ and ΔA has no isolated point then T is semi-Fredholm iff T is invertible.

Proof:

Let T be semi-Fredholm. By theorem 3,
$T \in \Phi_M(A) \Rightarrow \exists S \in M(A)$ and
$K \in K(A) \ni TS = ST = I + K$. K = 0 because ΔA contains no isolated points. Hence, $TS = ST = I$

Conversely,

If T is invertible then $TS = ST = I$ for some S in M(A) implies $T \in \Phi_M(A) \Rightarrow$ T is semi-Fredholm [Theorem 3].

REFERENCES

[1] S. Helgason, 1956. Multipliers of Banach Algebra. Ann. of Math., 64 (2): 240-254.

[2] F.T. Birtel, 1962. Isomorphisms and Isometric Multipliers. Proc. Amer. Math. Soc., 13: 204-210.

[3] J.K.Wang, 1961. Multipliers of Commutative Banach Algebras. Pacific J. Math., 11: 1131-1149.

[4] R.Larsen, 1971. An Introduction to the Theory of Multipliers. Die Grundlehren der Mathematischen Wissenschaften, Springer-Verlage, NY, Heidelberg, Vol: 75.

[5] T. Husain, 1989. Multipliers on Topological Algebras (ISBN 8301090383), Panstwowe Wydewn Nauk.

[6] M. Noor, 1995. Spectral Properties of Multipliers on Topological Algebras, Mathematical Reports, Acad. Sc. Canada, 17 (6): 238-242.

[7] M. Noor, 1996. On Compact Multipliers of Topological Algebras. Periodica Mathematica Hungarica, 32 (3): 209-212.

[8] L.A. Khan, M. Noor & A.B. Thaheem,1999. Double Multipliers on Topological Algebras. Internat. J. Math. Math Sci., 22 (3): 629-636.

[9] H. Render and A. Sauer, 2000. Multipliers on Vector Spaces of Holomorphic Function. Nagoya Math J., 159: 167-178.

[10] M. Azram, 2006. Multipliers on commutative Semi-simple Fre'chet Algebra. J. Sci. Tech. Uni. Pesh,30 (1): 15-17.

[11] M. Azram and A. Shehla, 2013. Multipliers on Fréchet Algebra, Middle-East Journal of Scientific Research 13 (Mathematical Applications in Engineering): 77-82.

[12] M. Azram, 2016. Topological Algebra via Inner Product, MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES, 10(S): 49-54.

[13] D.V.Dantzig,1931. Studi & &enover Topologische Algebra, Ph.D. Thesis, Gro& & ningen.

[14] R.F. Arens, 1947. Pseudo-normed Algebra. Bull. Amer. Math. Soc. 53.

[15] E.A.Michael, 1952. Locally Multiplicatively-convex Topological Algebra, Mem. Amer. Math. Soc.11

[16] H.G. Heuser, 1982. Functional Analysis. John Wiley and Sons Ltd., NY.

[17] W. Rudin, 1981. Functional Analysis. Tata McGraw Hill Publishing Co.Ltd., New Delhi.